Quasi, twisted, and all that... in Poisson geometry and Lie algebroid theory

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Dedicated to Alan Weinstein

Abstract

Motivated by questions from quantum group and field theories, we review structures on manifolds that are weaker versions of Poisson structures, and variants of the notion of Lie algebroid. We give a simple definition of the Courant algebroids and introduce the notion of a deriving operator for the Courant bracket of the double of a proto-bialgebroid. We then describe and relate the various quasi-Poisson structures, which have appeared in the literature since 1991, and the twisted Poisson structures studied by Ševera and Weinstein.

Introduction

In 1986, Drinfeld introduced both the *quasi-Hopf algebras*, that generalize the Hopf algebras defining quantum groups, and their semi-classical limits, the *Lie quasi-bialgebras*. This naturally led to the notion of *quasi-Poisson Lie groups* which I introduced in [27] [28].

A *quasi-Hopf algebra* is a bialgebra in which the multiplication is associative but the co-multiplication is only co-associative up to a defect measured by an element Φ in the triple tensor product of the algebra. Similarly, the definitions of the Lie quasi-bialgebras and the quasi-Poisson Lie groups involve a given element in $\bigwedge^3 \mathfrak{g}$, where $\mathfrak{g}$ is the underlying Lie algebra, which Drinfeld denoted by φ. In a Lie quasi-bialgebra, the bracket is a Lie bracket because it satisfies the Jacobi identity, but the compatible cobracket is not a true Lie bracket on the dual of $\mathfrak{g}$, because it only satisfies the Jacobi identity up to a defect measured by the element φ. On a quasi-Poisson Lie group, there is a multiplicative bivector field, π, whose Schouten bracket, $[\pi, \pi]$, does not vanish, but is also expressed in terms of φ. The desire to understand the *group-valued moment maps* and the *quasi-hamiltonian spaces* of Alekseev, Malkin and Meinrenken [3] in terms of Poisson geometry led to the study of the action of quasi-Poisson Lie groups on manifolds equipped with a bivector field [1]. A special case of a quasi-Poisson structure on a Lie group occurs when the bivector vanishes and only φ remains, corresponding to a Lie quasi-bialgebra with a trivial cobracket. The *quasi-Poisson manifolds* studied in [2] are manifolds equipped with a bivector, on which such a quasi-Poisson Lie group acts.

Recently, closed 3-form fields appeared in Park's work on string theory [42], and in the work on topological field theory of Klimčik and Strobl, who recognized the appearance of a new geometrical structure which they called WZW-Poisson manifolds [25]. They chose this name because the role of the background 3-form is analogous to that of the Wess-Zumino term introduced by Witten in a field theory with target a group, and more recently they proposed to shorten the name to WZ-Poisson manifolds. Shortly after these publications circulated as preprints, Ševera and Weinstein studied such structures in the framework of Courant algebroid theory, calling them *Poisson structures with a* 3*-form background*. They are defined in terms of a bivector field π and a closed 3-form, denoted by φ in [48], but which we shall denote by ψ to avoid confusion with the above. Again π is not a Poisson bivector – unless ψ vanishes, in which case the Poisson structure with background reduces to a Poisson structure –, its Schouten bracket is the image of the 3-form ψ under the morphism of vector bundles defined by π, mapping forms to vectors. Ševera and Weinstein also called the Poisson structures with background *ψ-Poisson structures*, or *twisted Poisson structures*. This last term has since been widely used [44][47][12][11], hence the word "twisted" in the title of this paper. It is justified by a related usage in the theory of "twisted sheaves", and we shall occasionally use this term but we prefer Poisson structure with background because, in Drinfeld's theory of Lie quasi-bialgebras, the words "twist"and "twisting" have a different and now standard meaning. Section 4.1 of this paper is a generalization of Drinfeld's theory to the Lie algebroid setting.

The theory of Lie bialgebras, on the one hand, is a special case of that of the Lie bialgebroids, introduced by Mackenzie and Xu [39]. It was shown by Roytenberg [44] that the "quasi" variant of this notion is the framework in which the Poisson structures with background appear naturally. Lie algebras, on the other hand, are a special case of the *Loday algebras*. Combining the two approaches, we encounter the *Courant algebroids* of Liu, Weinstein and Xu [35], or rather their equivalent definition in terms of non-skew-symmetric brackets.

We shall present these *a priori* different notions, and shall show how they can be related. In Section 1, we give a brief overview of the various theories just mentioned. In particular we define the *proto-bialgebroids* and the *Lie quasi-bialgebroids*, which generalize the Lie quasi-bialgebras, as well as their duals, the *quasi-Lie bialgebroids*. In Section 2, we give a simple definition of the Courant algebroids, which we prove to be equivalent to the usual definition [35] [43] (Theorem 2.1). Liu, Weinstein and Xu [35] showed that the construction of the double of Lie bialgebroids can be accomplished in the framework of Courant algebroid theory by introducing *Manin triples for Lie bialgebroids*. Along the lines of [44], we extend these considerations to the case of proto-bialgebroids and, in particular, to both "quasi" cases. Thus, we study the more general *Manin pairs for Lie quasi-bialgebroids*. This is the subject of Section 3, where we also introduce the notion of a *deriving operator* (in the spirit of [30] and [32]) for the double of a proto-bialgebroid, and we prove an existence theorem (Theorem 3.2). Section 4 is devoted to the study of examples. The *twisting* of Lie quasi-bialgebroids by bivectors generalizes Drinfeld's twisting of Lie quasi-bialgebras, and leads to the consideration of the quasi-Maurer-Cartan equation, which generalizes the quasi-Poisson condition. One can twist a quasi-Lie bialgebroid *with a closed* 3*-form background* by a bivector, and the Poisson condition with

background appears as the condition for the twisted object to remain a quasi-Lie bialgebroid.

The world of the "quasi" structures which we explore here is certainly nothing but a small part of the realm of homotopy structures, L_∞, G_∞, etc. See, in particular, [53] and the articles of Stasheff [49], Bangoura [7] and Huebschmann [21]. We hope to show that these are interesting objects in themselves.

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1 A review

Before we mention the global objects such as the generalizations of the Poisson Lie groups, we shall recall their infinitesimal counterparts.

1.1 Lie quasi-bialgebras, quasi-Lie bialgebras and proto-bialgebras

We shall not review all the details of the structures that are weaker versions of the Lie bialgebra structure, but we need to recall the definition of Lie quasi-bialgebras. It is due to Drinfeld [15], while in [28] and [8] the dual case, that of a quasi-Lie bialgebra, and the more general case of proto-bialgebras (called there "proto-Lie-bialgebras") are treated. A *proto-bialgebra* structure on a vector space F is defined by a quadruple of elements in $\bigwedge^\bullet(F \oplus F^*) \simeq C^\infty T^*(\Pi F)$, where Π denotes the change of parity. We denote such a quadruple by $(\mu, \gamma, \varphi, \psi)$, with $\mu : \bigwedge^2 F \to F$, $\gamma : \bigwedge^2 F^* \to F^*$, $\varphi \in \bigwedge^3 F$, $\psi \in \bigwedge^3 F^*$. This quadruple defines a proto-bialgebra if and only if $\{\mu+\gamma+\varphi+\psi, \mu+\gamma+\varphi+\psi\} = 0$, where $\{\ ,\ \}$ is the canonical Poisson bracket of the cotangent bundle $T^*(\Pi F)$, which coincides with the *big bracket* of $\bigwedge^\bullet(F \oplus F^*)$ [28]. This condition is equivalent to the five conditions which we shall write below in the more general case of the proto-bialgebroids (see Section 1.5). If either ψ or φ vanishes, there remain only four non-trivial conditions. When $\psi = 0$, the bracket is a Lie bracket, while the cobracket only satisfies the Jacobi identity up to a term involving φ, and we call the proto-bialgebra a *Lie quasi-bialgebra*. When $\varphi = 0$, the bracket only satisfies the Jacobi identity up to a term involving ψ, while the cobracket is a Lie cobracket, and we call the proto-bialgebra a *quasi-Lie bialgebra*. Clearly, the dual of a Lie quasi-bialgebra is a quasi-Lie bialgebra, and conversely.

Drinfeld only considered the case $\psi = 0$. In the English translation of [15], what we call a Lie quasi-bialgebra in this paper was translated as a quasi-Lie bialgebra, a term which we shall reserve for the object *dual* to a Lie quasi-bialgebra. In fact, it is in the dual object, where $\varphi = 0$ and $\psi \neq 0$ that the algebra structure is only "quasi-Lie". As another potential source of confusion, we mention that in [43] and [44], the element in $\bigwedge^3 F^*$ that we denote by ψ is denoted by φ, and vice-versa.

Any proto-bialgebra $((F, F^*), \mu, \gamma, \varphi, \psi)$ has a *double* which is $\mathfrak{d} = F \oplus F^*$, with

the Lie bracket,

$$\begin{aligned}
[x,y] &= \mu(x,y) + i_{x\wedge y}\psi \ , \\
[x,\xi] &= -[\xi,x] = -ad^{*\gamma}_{\xi}x + ad^{*\mu}_{x}\xi \ , \\
[\xi,\eta] &= i_{\xi\wedge\eta}\varphi + \gamma(\xi,\eta) \ .
\end{aligned}$$

Here x and $y \in F$, and ξ and $\eta \in F^*$.

Any Lie bialgebra has, associated with it, a pair of Batalin-Vilkovisky algebras in duality. The extension of this property to Lie quasi-bialgebras, giving rise to quasi-Batalin-Vilkovisky algebras in the sense of Getzler [18], has been carried out by Bangoura [5]. There is a notion of quasi-Gerstenhaber algebra (see [44]), and Bangoura has further proved that quasi-Batalin-Vilkovisky algebras give rise to quasi-Gerstenhaber algebras [6]. For a thorough study of these notions in the general algebraic setting, see Huebschmann [21]. These "quasi" algebras are the simplest examples of G_∞- and BV_∞-algebras, in which all the higher-order multilinear maps vanish except for the trilinear map.

1.2 Quasi-Poisson Lie groups and moment maps with values in homogeneous spaces

The global object corresponding to the Lie quasi-bialgebras we have just presented was introduced in [28] and called a *quasi-Poisson Lie group*. It is a Lie group with a multiplicative bivector, π_G, whose Schouten bracket does not vanish (so that it is not a Poisson bivector), but is a coboundary, namely

$$\frac{1}{2}[\pi_G,\pi_G] = \varphi^L - \varphi^R \ ,$$

where φ^L (resp., φ^R) are the left- (resp., right-)invariant trivectors on the group with value $\varphi \in \bigwedge^3 \mathfrak{g}$ at the identity. In [1], we considered the action of a quasi-Poisson Lie group (G,π_G,φ) on a manifold M equipped with a G-invariant bivector π. When the Schouten bracket of π satisfies the condition

$$\frac{1}{2}[\pi,\pi] = \varphi_M \ , \tag{1.1}$$

we say that (M,π) is a *quasi-Poisson G-space*. Here φ_M is the image of the element φ in $\bigwedge^3\mathfrak{g}$ under the infinitesimal action of the Lie algebra $\mathfrak{g}$ of G on M. The quasi-Poisson G-space (M,π) is called a *hamiltonian quasi-Poisson G-space* if there exists a moment map for the action of G on M, which takes values in D/G, where D is the simply connected Lie group whose Lie algebra is the double $\mathfrak{d} = \mathfrak{g}\oplus\mathfrak{g}^*$ of the Lie quasi-bialgebra $\mathfrak{g}$. See [1] for the precise definitions.

Two extreme cases of this construction are of particular interest. The first corresponds to the case where the Lie quasi-bialgebra is actually a Lie bialgebra ($\varphi = 0$), *i.e.*, the Manin pair with a chosen isotropic complement defining the Lie quasi-bialgebra is in fact a Manin triple. Then G is a Poisson Lie group and D/G can be identified with a dual group G^* of G. The moment maps for the quasi-hamiltonian G-spaces reduce to the moment maps in the sense of Lu [37] that take values in the dual Poisson Lie group, G^*. The second case is that of a Lie quasi-bialgebra with vanishing cobracket ($\gamma = 0$), to be described in the next subsection.

1.3 Quasi-Poisson manifolds and group-valued moment maps

Assume that G is a Lie group acting on a manifold M, and that $\mathfrak{g}$ is a *quadratic Lie algebra*, *i.e.*, a Lie algebra with an invariant non-degenerate symmetric bilinear form. We consider the bilinear form in $\mathfrak{g} \oplus \mathfrak{g}$ defined as the difference of the copies of the given bilinear form on the two terms of the direct sum. Let $\mathfrak{g}$ be diagonally embedded into $\mathfrak{g} \oplus \mathfrak{g}$. Then $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g})$ is a Manin pair, and we choose the anti-diagonal, $\{(x, -x) | x \in \mathfrak{g}\}$, as a complement of $\mathfrak{g} \subset \mathfrak{g} \oplus \mathfrak{g}$. The corresponding Lie quasi-bialgebra has vanishing cobracket, because the bracket of two elements in the anti-diagonal is in the diagonal, and therefore the bivector of the quasi-Poisson structure of G is trivial. With this choice of a complement, φ is the Cartan trivector of $\mathfrak{g}$. In this way, we obtain the *quasi-Poisson G-manifolds* described in [2]. They are pairs, (M, π), where π is a G-invariant bivector on M that satisfies equation (1.1) with φ the Cartan trivector of $\mathfrak{g}$. The group G acting on itself by means of the adjoint action is a quasi-Poisson G-manifold, and so are its conjugacy classes. The bivector π_G on G is $\sum_a e_a^R \wedge e_a^L$, where e_a is an orthonormal basis of $\mathfrak{g}$. Because the homogeneous space D/G of the general theory is the group G itself in this case, the moment maps for the *hamiltonian quasi-Poisson manifolds* are group-valued. Those hamiltonian quasi-Poisson manifolds for which the bivector π satisfies a non-degeneracy condition are precisely the *quasi-hamiltonian manifolds* of Alekseev, Malkin and Meinrenken [3].

1.4 Lie bialgebroids and their doubles

Lie bialgebroids were first defined by Mackenzie and Xu [39]. We state the definition as we reformulated it in [29]. To each *Lie algebroid* A are associated

- a Gerstenhaber bracket, $[\ ,\ ]_A$, on $\Gamma(\bigwedge^\bullet A)$,
- a differential, d_A, on $\Gamma(\bigwedge^\bullet A^*)$.

A *Lie bialgebroid* is a pair, (A, A^*), of Lie algebroids in duality such that d_{A^*} is a derivation of $[\ ,\ ]_A$, or, equivalently, d_A is a derivation of $[\ ,\ ]_{A^*}$.

Extending the construction of the Drinfeld double of a Lie bialgebra to the case of a Lie bialgebroid is a non-trivial problem, and several solutions have been offered, by Liu, Weinstein and Xu [35] in terms of the Courant algebroid $A \oplus A^*$, by Mackenzie [38] in terms of the double vector bundle $T^*A \simeq T^*A^*$, and by Vaintrob (unpublished) and Roytenberg [43] [44] in terms of supermanifolds. We shall describe some properties of the first and third constructions in Section 3.

1.5 Lie quasi-bialgebroids, quasi-Lie bialgebroids, proto-bialgebroids and their doubles

We call attention to the fact that we shall define here both "Lie quasi-bialgebroids" and "quasi-Lie bialgebroids" and that, as we explain below, these terms are not synonymous. We extend the notations of [15], [28], [8] to the case of Lie algebroids.

A *proto-bialgebroid* (A, A^*) is defined by anchors ρ_A and ρ_{A^*}, brackets $[\ ,\ ]_A$ and $[\ ,\ ]_{A^*}$, and elements $\varphi \in \Gamma(\bigwedge^3 A)$ and $\psi \in \Gamma(\bigwedge^3 A^*)$. By definition,

- The case $\psi = 0$ is that of *Lie quasi-bialgebroids* (A is a true Lie algebroid, while A^* is only "quasi"),
- The case $\varphi = 0$ is that of *quasi-Lie bialgebroids* (A^* is a true Lie algebroid, while A is only "quasi").
- The case where both φ and ψ vanish is that of the *Lie bialgebroids.*

While the dual of a Lie bialgebroid is itself a Lie bialgebroid, the dual of a Lie quasi-bialgebroid is a quasi-Lie bialgebroid, and conversely.

Whenever A is a vector bundle, the space of functions on $T^*\Pi A$, where Π denotes the change of parity, contains the space of sections of $\bigwedge^\bullet A$, the A-multivectors. In particular, the sections of A can be considered as functions on $T^*\Pi A$. Given the canonical isomorphism, $T^*\Pi A^* \simeq T^*\Pi A$, the same conclusion holds for the sections of $\bigwedge^\bullet A^*$, in particular for the sections of A^*.

A *Lie algebroid bracket* $[\ ,\ ]_A$ on a vector bundle A over a manifold M is defined, together with an anchor $\rho_A\ : A \to TM$, by a function μ on the supermanifold $T^*\Pi A$ ([43] [44] [51] [52]). Let $\{\ ,\ \}$ denote the canonical Poisson bracket of the cotangent bundle. The bracket of two sections x and y of A is the *derived bracket*, in the sense of [30],

$$[x, y]_A = \{\{x, \mu\}, y\}\ ,$$

and the anchor satisfies

$$\rho_A(x)f = \{\{x, \mu\}, f\}\ ,$$

for $f \in C^\infty(M)$. When (A, A^*) is a pair of Lie algebroids in duality, both $[\ ,\ ]_A$ together with ρ_A, and $[\ ,\ ]_{A^*}$ together with ρ_{A^*} correspond to functions, denoted by μ and γ, on the same supermanifold $T^*\Pi A$, taking into account the identification of $T^*\Pi A^*$ with $T^*\Pi A$. The three conditions in the definition of a *Lie bialgebroid* are equivalent to the single equation

$$\{\mu + \gamma, \mu + \gamma\} = 0\ .$$

More generally, the five conditions for a *proto-bialgebroid* defined by $(\mu, \gamma, \varphi, \psi)$ are obtained from a single equation. By definition, a *proto-bialgebroid* structure on (A, A^*) is a function of degree 3 and of Poisson square 0 on $T^*\Pi A$. As in the case of a proto-bialgebra, such a function can be written $\mu + \gamma + \varphi + \psi$, where $\varphi \in \Gamma \bigwedge^3 A$ and $\psi \in \Gamma \bigwedge^3 A^*$, and satisfies

$$\{\mu + \gamma + \varphi + \psi, \mu + \gamma + \varphi + \psi\} = 0\ . \tag{1.2}$$

The definition is equivalent to the conditions

$$\left\{ \begin{array}{rcl} \frac{1}{2}\{\mu, \mu\} + \{\gamma, \psi\} & = & 0\ , \\ \{\mu, \gamma\} + \{\varphi, \psi\} & = & 0\ , \\ \frac{1}{2}\{\gamma, \gamma\} + \{\mu, \varphi\} & = & 0\ , \\ \{\mu, \psi\} & = & 0\ , \\ \{\gamma, \varphi\} & = & 0\ . \end{array} \right.$$

- When $((A, A^*), \mu, \gamma, \varphi, 0)$ is a Lie quasi-bialgebroid, $((A, A^*), \mu, \gamma)$ is a Lie bialgebroid if and only if $\{\mu, \varphi\} = 0$.
- Dually, when $((A, A^*), \mu, \gamma, 0, \psi)$ is a quasi-Lie bialgebroid, $((A, A^*), \mu, \gamma)$ is a Lie bialgebroid if and only if $\{\gamma, \psi\} = 0$.

Remark In the case of a proto-bialgebra, (F, F^*), the operator $\{\mu, \ .\}$ generalizes the Chevalley-Eilenberg coboundary operator on cochains on F with values in $\bigwedge^\bullet F$. In the term $\{\mu, \varphi\}$, φ should be viewed as a 0-cochain on F with values in $\bigwedge^3 F$, and $\{\mu, \varphi\}$ is an element in $F^* \otimes \bigwedge^3 F$. So is $\{\gamma, \gamma\}$, which is a trilinear form on F^* with values in F^* whose vanishing is equivalent to the Jacobi identity for γ. In the term $\{\mu, \psi\}$, ψ should be viewed as a 3-cochain on F with scalar values, and $\{\mu, \psi\}$ is an element in $\bigwedge^4 F^*$. Reversing the roles of F and F^*, one obtains the interpretation of the other terms in the above formulas.

1.6 Poisson structures with background (twisted Poisson structures)

The WZW-Poisson structures introduced by Klimčik and Strobl [25] were studied by Ševera and Weinstein in 2001 [48], who called them *Poisson structures with background*, and also *twisted Poisson structures*. Roytenberg has subsequently shown that they appear by a twisting of a quasi-Lie bialgebroid by a bivector [44]. We shall review this approach in Section 4. The integration of Poisson structures with background into *quasi-symplectic groupoids* is the subject of recent work of Bursztyn, Crainic, Weinstein and Zhu [11] and of Cattaneo and Xu [12]. In addition, Xu [55] has very recently extended the theory of momentum maps to this setting.

1.7 Other structures: Loday algebras, omni-Lie algebras

There are essentially two ways of weakening the properties of Lie algebras. One possibility is to introduce a weakened version of the Jacobi identity, *e.g.*, an identity up to homotopy: this is the theory of L_∞-algebras. The relationship of the Courant algebroids to L_∞-algebras was explored in [46].

Another possibility is to consider non-skew-symmetric brackets: this is the theory of *Loday algebras*, which Loday introduced and called *Leibniz algebras*. A Loday algebra is a graded vector space with a bilinear bracket of degree n satisfying the Jacobi identity,

$$[x, [y, z]] = [[x, y], z] + (-1)^{(|x|+n)(|y|+n)}[y, [x, z]] \ , \tag{1.3}$$

for all elements x, y and z, where $|x|$ is the degree of x. In Section 2, we shall describe the Loday algebra approach to Courant algebroids, in which case there is no grading.

The "omni-Lie algebras" introduced by Weinstein in [54] provide an elegant way of characterizing the Lie algebra structures on a vector space V in terms of the graph in $V \times \mathfrak{gl}(V)$ of the adjoint operator. In the same paper, he defined the $(R, \mathcal{A})$ C-algebras, the algebraic analogue of Courant algebroids, which generalize the $(R, \mathcal{A})$ Lie algebras (also called Lie-Rinehart algebras or pseudo-Lie algebras), and he posed the question of how to determine the global analogue of an "omni-Lie algebra". In [24], he and Kinyon explored this problem and initiated the search for the global objects associated to generalized Lie algebras, that would generalize Lie groups. They proved new properties of the Loday algebras, showing in what sense they can be integrated to a homogeneous left loop, *i.e.*, to a manifold with a non-associative

composition law, and they showed that the Courant brackets of the doubles of Lie bialgebroids can be realized on the tangent spaces of reductive homogeneous spaces. These global constructions are inspired by the correspondence between generalized Lie triple systems and non-associative multiplications on homogeneous spaces. (Some of the results of Bertram [9] might prove useful in the search for global objects integrating generalized Lie algebras.) For recent developments, see Kinyon's lecture [23].

1.8 Generalized Poisson brackets for non-holonomic mechanical systems

Brackets of the Poisson or Dirac type that do not satisfy the Jacobi identity appear in many geometric constructions describing non-holonomic mechanical systems. There is a large literature on the subject; see for instance [22] [13] and their references. It would be very interesting to study how these constructions relate to the various structures which we are now considering. In his lecture [38], Marsden showed how to state the non-holonomic equations of Lagrangian mechanics in terms of isotropic subbundles in the direct sum of the tangent and cotangent bundles of the phase space, T^*Q, of the system under consideration. He calls such subbundles Dirac structures on T^*Q. Yet, it is only when an integrability condition is required that these structures become examples of the Dirac structures to be mentioned in the next section.

2 Courant algebroids

The construction of the double of a Lie bialgebra with the structure of a Lie algebra does not extend into a construction of the double of a Lie bialgebroid with the structure of a Lie algebroid, because the framework of Lie algebroid theory is too narrow to permit it. While it is not the only solution available, the introduction of the new notion of *Courant algebroid* permits the solution of this problem.

The definition of Courant algebroids, based on Courant's earlier work [14], is due to Liu, Weinstein and Xu [35]. It was shown by Roytenberg [43] that a Courant algebroid can be equivalently defined as a vector bundle $E \to M$ with a Loday bracket on ΓE, an anchor $\rho\ : E \to TM$ and a field of non-degenerate, symmetric bilinear forms $(\ |\ )$ on the fibers of E, related by a set of four additional properties. It was further observed by Uchino [50] and by Grabowski and Marmo [19] that the number of independent conditions can be reduced. We now show that it can be reduced to two properties which are very natural generalizations of those of a quadratic Lie algebra. In fact, (i) and (ii) below are generalizations to algebroids of the skew-symmetry of the Lie bracket, and of the condition of ad-invariance for a bilinear form on a Lie algebra, respectively.

Definition 2.1 *A* Courant algebroid *is a vector bundle $E \to M$ with a Loday bracket on ΓE, i.e., an $\mathbb{R}$-bilinear map satisfying the Jacobi identity,*

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]] \ ,$$

for all $x, y, z \in \Gamma E$, an anchor, $\rho : E \to TM$, which is a morphism of vector bundles, and a field of non-degenerate symmetric bilinear forms $(\ |\ )$ on the fibers of E, satisfying

$$\begin{array}{ll} \text{(i)} & \rho(x)(u|v) = (x \mid [u,v] + [v,u]) \ , \\ \text{(ii)} & \rho(x)(u|v) = ([x,u] \mid v) + (u \mid [x,v]) \ , \end{array}$$

for all x, u and $v \in \Gamma E$.

Remark Property (i) is equivalent to

$$\text{(i}'\text{)} \qquad \frac{1}{2}\rho(x)(y|y) = (x|[y,y])$$

(which is property 4 of Definition 2.6.1 in [43], and property 5 of Section 1 in [48]). The conjunction of properties (i) and (ii) is equivalent to property (ii) together with

$$\text{(i}''\text{)} \qquad (x|[y,y]) = ([x,y]|y)$$

(which is property 5 in Appendix A in [47]).

We now prove two important consequences of properties (i) and (ii) which have been initially considered to be additional, independent defining properties of Courant algebroids.

Theorem 2.1 *In any Courant algebroid,*
(iii) *the Leibniz rule is satisfied, i.e.,*

$$[x, fy] = f[x,y] + (\rho(x)f)y \ ,$$

for all x and $y \in \Gamma E$ and all $f \in C^\infty(M)$,
(iv) *the anchor, ρ, induces a morphism of Loday algebras from ΓE to $\Gamma(TM)$, i.e., it satisfies*

$$\rho([x,y]) = [\rho x, \rho y] \ ,$$

for all x and $y \in \Gamma E$.

Proof The proof of (iii), adapted from [50], is obtained by evaluating $\rho(x)(fy|z)$ in two ways. We first write, using the Leibniz rule for vector fields acting on functions,

$$\rho(x)(fy|z) = (\rho(x)f)(y|z) + f\rho(x)(y|z) \ .$$

Then, using property (ii) twice, we obtain

$$([x,fy]|z) + (fy|[x,z]) = (\rho(x)f)(y|z) + f([x,y]|z) + f(y|[x,z]) \ .$$

and (iii) follows by the non-degeneracy of $(\ |\ )$.

The proof of (iv) is that of the analogous property for Lie algebroids (see, *e.g.*, [33]). It is obtained by evaluating $[x,[y,fz]]$, for $z \in \Gamma E$, in two ways, using both the Jacobi identity for the Loday bracket $[\ ,\ ]$ and (iii). □

It follows from the Remark together with Theorem 2.1 and from the arguments of Roytenberg in [43] that our definition of Courant algebroids is equivalent to that of Liu, Weinstein and Xu in [35].

A *Dirac sub-bundle* (also called a *Dirac structure*) in a Courant algebroid is a maximally isotropic sub-bundle whose space of sections is closed under the bracket.

Courant algebroids with base a point are *quadratic Lie algebras*. More generally, Courant algebroids with a trivial anchor are bundles of quadratic Lie algebras with a smoothly varying structure.

The notion of a Dirac sub-bundle in a Courant algebroid with base a point reduces to that of a maximally isotropic Lie subalgebra in a quadratic Lie algebra, in other words, to a *Manin pair*. We shall show that a Courant algebroid together with a Dirac sub-bundle is an appropriate generalization of the notion of a Manin pair from the setting of Lie algebras to that of Lie algebroids.

A deep understanding of the nature of Courant algebroids is provided by the consideration of the *non-negatively graded manifolds*. This notion was defined and used by Kontsevich [26], Ševera [47] (who called them N*-manifolds*) and T. Voronov [52]. In [45], Roytenberg showed that the non-negatively graded symplectic manifolds of degree 2 are the pseudo-euclidian vector bundles, and that the Courant algebroids are defined by an additional structure, that of a homological vector field, associated to a cubic hamiltonian Θ of Poisson square 0, preserving the symplectic structure. The bracket and the anchor of the Courant algebroid are recovered from this data as the derived brackets, $[x, y] = \{\{x, \Theta\}, y\}$ and $\rho(x)f = \{\{x, \Theta\}, f\}$. T. Voronov [52] studied the double of the non-negatively graded QP-manifolds which are a generalization of the Lie bialgebroids.

3 The double of a proto-bialgebroid

We shall now explain how to generalize the construction of a double with a Courant algebroid structure from Lie bialgebroids to proto-bialgebroids.

3.1 The double of a Lie bialgebroid

Liu, Weinstein and Xu [35] have shown that complementary pairs of Dirac sub-bundles in a Courant algebroid are in one-to-one correspondence with Lie bialgebroids:

If E is a Courant algebroid, if $E = A \oplus B$, where A and B are maximally isotropic sub-bundles, and if ΓA and ΓB are closed under the bracket, then

- A and B are in duality, $B \simeq A^*$,
- the bracket of E induces Lie algebroid brackets on A and $B \simeq A^*$, with respective anchors the restrictions of the anchor of E to A and A^*,
- the pair (A, A^*) is a Lie bialgebroid.

Conversely, if (A, A^*) is a Lie bialgebroid, the direct sum $A \oplus A^*$ is equipped with a Courant algebroid structure such that A and A^* are maximally isotropic sub-bundles, and ΓA and $\Gamma(A^*)$ are closed under the bracket, the bilinear form being the canonical one, defined by

$$(x + \xi | y + \eta) = < \xi, y > + < \eta, x > ,$$

for x and $y \in \Gamma A$, ξ and $\eta \in \Gamma(A^*)$.

3.2 The case of proto-bialgebroids

The construction which we just recalled can be extended to the proto-bialgebroids [44]. Let A be a vector bundle. Recall that a proto-bialgebroid structure on (A, A^*) is a function of degree 3 and of Poisson square 0 on $T^*\Pi A$, that can be written $\mu + \gamma + \varphi + \psi$, where $\varphi \in \Gamma(\bigwedge^3 A)$ and $\psi \in \Gamma(\bigwedge^3 A^*)$, and μ (resp., γ) defines a bracket and anchor on A (resp., A^*).

The Courant bracket of the double, $A \oplus A^*$, of a proto-bialgebroid, (A, A^*), defined by $(\mu, \gamma, \varphi, , \psi)$, is the *derived bracket*,

$$[x + \xi, y + \eta] = \{\{x + \xi, \mu + \gamma + \varphi + \psi\}, y + \eta\} \ .$$

Here x and y are sections of A, ξ and η are sections of A^*, and $[x + \xi, y + \eta]$ is a section of $A \oplus A^*$. (The right-hand side makes sense more generally when x and y are A-multivectors, and ξ and η are A^*-multivectors, but the resulting quantity is not necessarily a section of $\bigwedge^\bullet(A \oplus A^*)$.)

The anchor is defined by

$$\{\{x + \xi, \mu + \gamma + \varphi + \psi\}, f\} = \{\{x, \mu\}, f\} + \{\{\xi, \gamma\}, f\} = (\rho_A(x) + \rho_{A^*}(\xi))(f) \ ,$$

for $f \in C^\infty(M)$. We set $[x, y]_\mu = \{\{x, \mu\}, y\}$ and $[\xi, \eta]_\gamma = \{\{\xi, \gamma\}, \eta\}$. The associated *quasi-differentials*, d_μ and d_γ, on $\Gamma(\bigwedge^\bullet A^*)$ and $\Gamma(\bigwedge^\bullet A)$ are

$$d_\mu = \{\mu, \ \cdot \ \} \qquad \text{and} \qquad d_\gamma = \{\gamma, \ \cdot \ \} \ ,$$

which satisfy

$$(d_\mu)^2 + \{d_\gamma \psi, \ \cdot \ \} = 0 \ , \qquad (d_\gamma)^2 + \{d_\mu \varphi, \ \cdot \ \} = 0 \ .$$

We denote the interior product of a form α by a multivector x by $i_x \alpha$, with the sign convention,

$$i_{x \wedge y} = i_x \circ i_y \ ,$$

and we use an analogous notation for the interior product of a multivector by a form. The Lie derivations are defined by $L^\mu_x = [i_x, d_\mu]$ and $L^\gamma_\xi = [i_\xi, d_\gamma]$. We find, for x and $y \in \Gamma A$, ξ and $\eta \in \Gamma(A^*)$,

$$[x, y] = [x, y]_\mu + i_{x \wedge y} \psi \ , \tag{3.1}$$

$$[x, \xi] = -i_\xi d_\gamma x + L^\mu_x \xi \ , \tag{3.2}$$

$$[\xi, x] = L^\gamma_\xi x - i_x d_\mu \xi \ , \tag{3.3}$$

$$[\xi, \eta] = i_{\xi \wedge \eta} \varphi + [\xi, \eta]_\gamma \ , \tag{3.4}$$

that is,

$$[x + \xi, y + \eta] = [x, y]_\mu + L^\gamma_\xi y - i_\eta d_\gamma x + i_{\xi \wedge \eta} \varphi + [\xi, \eta]_\gamma + L^\mu_x \eta - i_y d_\mu \xi + i_{x \wedge y} \psi \ .$$

These formulas extend both the Lie bracket of the Drinfeld double of a proto-bialgebra [8], recalled in Section 1.1, and the Courant bracket of the double of a Lie bialgebroid [35].

3.3 Deriving operators

If ξ is a section of A^*, by e_ξ we denote the operation of exterior multiplication by ξ on $\Gamma(\bigwedge^\bullet A^*)$. In this subsection, the square brackets $[\ ,\ ]$ without a subscript denote the graded commutators of endomorphisms of $\Gamma(\bigwedge^\bullet A^*)$.

Definition 3.1 *We say that a differential operator $\mathcal{D}$ on $\Gamma(\bigwedge^\bullet A^*)$ is a* deriving operator *for the Courant bracket of $A \oplus A^*$ if it satisfies the following relations,*

$$[[i_x, \mathcal{D}], i_y] = i_{[x,y]_\mu} + e_{i_{x\wedge y}\psi} \ , \tag{3.5}$$

$$[[i_x, \mathcal{D}], e_\xi] = -i_{i_\xi d_\gamma x} + e_{L^\mu_x \xi} \ , \tag{3.6}$$

$$[[e_\xi, \mathcal{D}], i_x] = i_{L^\gamma_\xi x} - e_{i_x d_\mu \xi} \ , \tag{3.7}$$

$$[[e_\xi, \mathcal{D}], e_\eta] = i_{i_{\xi\wedge\eta}\varphi} + e_{[\xi,\eta]_\gamma} \ . \tag{3.8}$$

If we identify $x \in \Gamma A$ with $i_x \in \mathrm{End}(\Gamma(\bigwedge^\bullet \mathrm{A}^*))$, and $\xi \in \Gamma(A^*)$ with $e_\xi \in \mathrm{End}(\Gamma(\bigwedge^\bullet \mathrm{A}^*))$, the preceding relations become

$$\begin{aligned}
&[[x, \mathcal{D}], y] = [x, y]_\mu + i_{x\wedge y}\psi \ , \\
&[[x, \mathcal{D}], \xi] = -i_\xi d_\gamma x + L^\mu_x \xi \ , \\
&[[\xi, \mathcal{D}], x] = L^\gamma_\xi x - i_x d_\mu \xi \ , \\
&[[\xi, \mathcal{D}], \eta] = i_{\xi\wedge\eta}\varphi + [\xi, \eta]_\gamma \ ,
\end{aligned}$$

so that the Courant bracket defined in Section 3.2 can also be written as a derived bracket [30] [32].

Remark With the preceding identification, the relation $i_x e_\xi + e_\xi i_x = <\xi, x>$ implies that

$$(x + \xi)(y + \eta) + (y + \eta)(x + \xi) = (x + \xi | y + \eta) \ .$$

This shows that $\Gamma(\bigwedge^\bullet A^*)$ is a Clifford module of the Clifford bundle of $A \oplus A^*$, the point of departure of Alekseev and Xu in [4].

Does the Courant bracket of a proto-bialgebroid admit a deriving operator? We first treat the case of a Lie bialgebroid. The space $\Gamma(\bigwedge^\bullet A^*)$ has the structure of a Gerstenhaber algebra defined by γ. We shall assume that this Gerstenhaber algebra admits a *generator* in the following sense [34].

Definition 3.2 *Let $[\ ,\ ]_\mathcal{A}$ be any Gerstenhaber bracket on an associative, graded commutative algebra $(\mathcal{A}, \wedge)$. An operator, ∂, on $\mathcal{A}$ is a generator of the bracket if*

$$[u, v]_\mathcal{A} = (-1)^{|u|}(\partial(u \wedge v) - \partial u \wedge v - (-1)^{|u|} u \wedge \partial v) \ ,$$

for all u and $v \in \mathcal{A}$. In particular, a Batalin-Vilkovisky algebra is a Gerstenhaber algebra which admits a generator of square 0.

Lemma 3.1 *If ∂ is a generator of bracket $[\ ,\ ]_\mathcal{A}$, then, for all u and $v \in \mathcal{A}$,*

$$[e_u, \partial] = e_{\partial u} - [u,\ .\ ]_\mathcal{A} \ , \tag{3.9}$$

and

$$[[e_u, \partial], e_v] = -e_{[u,v]_\mathcal{A}} \ , \tag{3.10}$$

where e_u is left $\wedge$-multiplication by $u \in \mathcal{A}$.

Proof The first relation follows from the definitions by a short computation, and the second is a consequence of the first, since

$$[[e_u, \partial], e_v] = [e_{\partial u} - [u, \ \cdot \ ]_{\mathcal{A}}, e_v] = -[[u, \ \cdot \ ]_{\mathcal{A}}, e_v] = -e_{[u,v]_{\mathcal{A}}} \ ,$$

for all u and $v \in \mathcal{A}$. □

Theorem 3.1 *If ∂_* is a generator of the Gerstenhaber bracket of $\Gamma(\bigwedge^\bullet A^*)$, then $d_\mu - \partial_*$ is a deriving operator for the Courant bracket of $A \oplus A^*$.*

Proof We consider various operators acting on sections of $\bigwedge^\bullet A^*$. We recall from [31] (see [34] for the case $A = TM$) that, for any $x \in \Gamma A$,

$$[i_x, \partial_*] = -i_{d_\gamma x} \ . \tag{3.11}$$

We shall also make use of the following relations,

$$[e_\xi, d_\mu] = e_{d_\mu \xi} \ , \tag{3.12}$$

for $\xi \in A^*$,

$$[i_u, e_\xi] = (-1)^{|u|+1} i_{i_\xi u} \ , \tag{3.13}$$

for any $u \in \Gamma(\bigwedge^\bullet A)$, and

$$[e_\zeta, i_x] = (-1)^{|\zeta|+1} e_{i_x \zeta} \ , \tag{3.14}$$

for all $\zeta \in \Gamma(\bigwedge^\bullet A^*)$.

1) Let x and y be in ΓA. We compute $[i_x, d_\mu - \partial_*] = L^\mu_x + i_{d_\gamma x}$, whence

$$[[i_x, d_\mu - \partial_*], i_y] = i_{[x,y]_\mu} + [i_{d_\gamma x}, i_y] = i_{[x,y]_\mu} \ .$$

This proves (3.5), corresponding to (3.1).

2) Let x be in ΓA and let ξ be in $\Gamma(A^*)$. We compute

$$[[i_x, d_\mu - \partial_*], e_\xi] = [L^\mu_x, e_\xi] + [i_{d_\gamma x}, e_\xi] = e_{L^\mu_x \xi} - i_{i_\xi d_\gamma x} \ .$$

This proves (3.6), corresponding to (3.2).

3) Since ∂_* is a generating operator of $[\ ,\ ]_\gamma$, (3.9) is valid and therefore

$$[e_\xi, \partial_*] = e_{\partial_* \xi} - [\xi, \ \cdot \ ]_\gamma \ . \tag{3.15}$$

Since $\partial_* \xi$ is of degree 0, $e_{\partial_* \xi}$ commutes with i_x. Therefore

$$[[e_\xi, d_\mu - \partial_*], i_x] = [e_{d_\mu \xi}, i_x] - [e_{\partial_* \xi}, i_x] + [[\xi, \ \cdot \ ]_\gamma, i_x] = -e_{i_x d_\mu \xi} + [[\xi, \ \cdot \ ]_\gamma, i_x] \ .$$

Let us now prove that the derivation $[[\xi, \ \cdot \ ]_\gamma, i_x]$ of $\Gamma(\bigwedge^\bullet A^*)$ coincides with the derivation $i_{L^\gamma_\xi x}$. In fact, they both vanish on 0-forms, and on a 1-form α,

$$[\xi, i_x \alpha]_\gamma - i_x[\xi, \alpha]_\gamma = \ \rho_{A^*}(\xi) < \alpha, x > - < [\xi, \alpha]_\gamma, x > \ ,$$

while

$$i_{L^\gamma_\xi x}(\alpha) = < \alpha, L^\gamma_\xi x > = < \alpha, i_\xi d_\gamma x > + < \alpha, d_\gamma < \xi, x >>$$

$$= d_\gamma x(\xi, \alpha) + d_\gamma < \xi, x > (\alpha) = \rho_{A^*}(\xi) < \alpha, x > - < [\xi, \alpha]_\gamma, x > \ .$$

Thus

$$[[e_\xi, d_\mu - \partial_*], i_x] = -e_{i_x d_\mu \xi} + i_{L^\gamma_\xi x} \ ,$$

and (3.7), corresponding to (3.3), is proved.

4) Let ξ and η be sections of A^*. Then

$$[[e_\xi, d_\mu], e_\eta] = [e_{d_\mu \xi}, e_\eta] = 0 \ ,$$

while, by (3.10),

$$[[e_\xi, \partial_*], e_\eta] = -e_{[\xi,\eta]_\gamma} \ ,$$

proving (3.8), corresponding to (3.4). □

We now turn to the case of a proto-bialgebroid, defined by $\varphi \in \Gamma(\bigwedge^3 A)$ and $\psi \in \Gamma(\bigwedge^3 A^*)$. The additional terms in the four expressions to be evaluated are

1) $[[i_x, i_\varphi], i_y] = 0$, and $[[i_x, e_\psi], i_y] = [e_{i_x \psi}, i_y] = e_{i_{x \wedge y}\psi}$.

2) $[[i_x, i_\varphi], e_\xi] = 0$, and $[[i_x, e_\psi], e_\xi] = [e_{i_x \psi}, e_\xi] = 0$.

3) $[[e_\xi, i_\varphi], i_x] = [i_{i_\xi \varphi}, i_x] = 0$, and $[[e_\xi, e_\psi], i_x] = 0$.

4) $[[e_\xi, i_\varphi], e_\eta] = [i_{i_\xi \varphi}, e_\eta] = i_{i_{\xi \wedge \eta}\varphi}$, and $[[e_\xi, e_\psi], e_\eta] = 0$.

Therefore, we can generalize Theorem 3.1 as follows.

Theorem 3.2 *If ∂_* is a generator of the Gerstenhaber bracket of $\Gamma(\bigwedge^\bullet A^*)$, then $d_\mu - \partial_* + i_\varphi + e_\psi$ is a deriving operator for the Courant bracket of the double, $A \oplus A^*$, of the proto-bialgebroid (A, A^*) defined by $\varphi \in \Gamma(\bigwedge^3 A)$ and $\psi \in \Gamma(\bigwedge^3 A^*)$.*

It is clear that the addition to a deriving operator of derivations i_{x_0} and e_{ξ_0} of the associative, graded commutative algebra $\Gamma(\bigwedge^\bullet A^*)$ will furnish a new deriving operator. The importance of the notion of a deriving operator comes from the fact that, if we can modifiy d_μ and ∂_* by derivations of $\Gamma(\bigwedge^\bullet A^*)$ in such a way that the deriving operator has square 0, then the Jacobi identity for the resulting non-skew-symmetric bracket follows from the general properties of derived brackets that were proved in [30].

Let (A, μ) be a Lie algebroid, let (A, A^*) be the triangular Lie bialgebroid defined by a bivector $\pi \in \Gamma(\bigwedge^2 A)$ satisfying $[\pi, \pi]_\mu = 0$, and let $d_\pi = [\pi, \ . \ ]_\mu$ be the differential on $\Gamma(\bigwedge^\bullet A)$ (see Section 4.1.1 below). We assume that there exists a nowhere vanishing section, ν, of the top exterior power of the dual. Let ∂_ν be the generator of the Gerstenhaber bracket of $\Gamma(\bigwedge^\bullet A)$ defined by ν, which is a generator of square 0. We set

$$x_\nu = \partial_\nu \pi \ .$$

Then, x_ν is a section of A, which is called the *modular field* of (A, A^*) associated with ν [31]. We shall now give a short proof of the existence of a deriving operator of square 0 for the Courant bracket of the dual of (A, A^*).

Theorem 3.3 *The operator $d_\pi - \partial_\nu + e_{x_\nu}$ is a deriving operator of square 0 of the Courant bracket of the double of the Lie bialgebroid (A^*, A).*

Proof By definition, the Laplacian of the strong differential Batalin-Vilkovisky algebra $(\Gamma(\bigwedge^\bullet A), \partial_\nu, d_\pi)$ is $[d_\pi, \partial_\nu]$, and we know that it satisfies the relation

$$[d_\pi, \partial_\nu] = L^\mu_{x_\nu} \ .$$

(See [31], and [34] for the case of a Poisson manifold.) Since, by Theorem 3.2, the operator $d_\pi - \partial_\nu$ is a deriving operator, and since this property is not modified by the addition of the derivation e_{x_ν}, it is enough to prove that the operator $d_\pi - \partial_\nu + e_{x_\nu}$ is of square 0. In fact, since both d_π and ∂_ν are of square 0,

$$\frac{1}{2}[d_\pi - \partial_\nu, d_\pi - \partial_\nu] = -L^\mu_{x_\nu} = -[x_\nu, \ \cdot \ ]_\mu \ .$$

Therefore

$$\frac{1}{2}[d_\pi - \partial_\nu + e_{x_\nu}, d_\pi - \partial_\nu + e_{x_\nu}] = -[x_\nu, \ \cdot \ ]_\mu + [e_{x_\nu}, d_\pi] - [e_{x_\nu}, \partial_\nu] \ .$$

By (3.12), $[e_{x_\nu}, d_\pi] = e_{d_\pi x_\nu}$, which vanishes since x_ν leaves π invariant, while by (3.15), $[e_{x_\nu}, \partial_\nu] = e_{\partial_\nu x_\nu} - [x_\nu, \ \cdot \ ]_\mu$. In addition, $\partial_\nu x_\nu = 0$, since $x_\nu = \partial_\nu \pi$ and ∂_ν is of square 0. Therefore the square of $d_\pi - \partial_\nu + e_{x_\nu}$ vanishes. □

In particular, if (M, π) is a Poisson manifold, we obtain a deriving operator of square 0 of the Courant algebroid, double of the Lie bialgebroid (T^*M, TM), dual to the triangular Lie bialgebroid (TM, T^*M).

More generally, Alekseev and Xu [4] consider deriving operators of the Courant bracket of a Courant algebroid whose square is a scalar function, which they call "generating operators" (but which should not be confused with the generating operators of Batalin-Vilkovisky algebras). They show that there always exists such a generating operator for the double of a Lie bialgebroid, (A, A^*), and that its square is expressible in terms of the modular fields of A and A^* (see Theorem 5.1 and Corollary 5.9 of [4]). It is easily seen that the case of a triangular Lie bialgebroid is a particular case of their theorem and corollary, in which the generating operator is equal to the deriving operator of Theorem 3.3, and the square of the generating operator actually vanishes. In fact, in the case of a triangular Lie bialgebroid (A, A^*), the Laplacian $[d_\mu, \partial_\pi]$ of the strong differential Batalin-Vilkovisky algebra $(\Gamma(\bigwedge^\bullet A^*), \partial_\pi, d_\mu)$ vanishes because $\partial_\pi = [i_\pi, d_\mu]$, and therefore the modular field of A vanishes. In addition, $x_\nu = \frac{1}{2}X_0$, where X_0 is the modular field of A^* [16] and $\partial_\nu x_\nu = 0$. Hence, in the expression for the square of the generating operator given in [4], both terms vanish.

4 Examples

We shall first analyze various constructions of Lie bialgebroids, Lie quasi-bialgebroids and quasi-Lie bialgebroids, then we shall consider the Courant brackets in the theory of Poisson structures with background.

4.1 Twisting by a bivector

4.1.1 Triangular Lie bialgebroids

Let (A, μ) be a Lie algebroid, and let π be a section of $\bigwedge^2 A$. On the one hand, such sections generalize the *r-matrices* and *twists* of Lie bialgebra theory, and on the other hand, when $A = TM$, such sections are *bivector fields* on the manifold M. By extension, a section of $\bigwedge^2 A$ is called an *A-bivector*, or simply a *bivector*.

Let $\pi^\sharp$ be the vector bundle map from $\Gamma(A^*)$ to ΓA defined by $\pi^\sharp(\xi) = i_\xi \pi$, for $\xi \in \Gamma(A^*)$. Consider the bracket on A^* depending on both μ and π defined by

$$[\xi, \eta]_{\mu,\pi} = L^\mu_{\pi^\sharp\xi}\eta - L^\mu_{\pi^\sharp\eta}\xi - d_\mu(\pi(\xi,\eta)) \ , \tag{4.1}$$

for ξ and $\eta \in \Gamma(A^*)$. The following relation generalizes the equation $\gamma = -d_\mu r$ which is valid in a coboundary Lie bialgebra.

Theorem 4.1 *Set*

$$\gamma_{\mu,\pi} = \{\pi, \mu\} = -\{\mu, \pi\} \ . \tag{4.2}$$

Then
(i) *the associated quasi-differential on* $\Gamma(\bigwedge^\bullet A)$ *is*

$$d_\pi = [\pi, \ \cdot \ ]_\mu \ ; \tag{4.3}$$

(ii) *bracket* $[\xi, \eta]_{\mu,\pi}$, *defined by formula* (4.1), *is equal to the derived bracket,*

$$\{\{\xi, \gamma_{\mu,\pi}\}, \eta\} \ ;$$

(iii) *if, in addition,*

$$\varphi = -\frac{1}{2}[\pi, \pi]_\mu \ , \tag{4.4}$$

then $((A, A^*), \mu, \gamma_{\mu,\pi}, \varphi, 0)$ *is a Lie quasi-bialgebroid.*

Proof The proof of (i) is a straightforward application of the Jacobi identity. To prove (ii) it suffices to prove that the quasi-differential d_π is given by the usual Cartan formula in terms of the anchor $\pi^\sharp$ and the Koszul bracket (4.1). This now classic result was first proved by Bhaskara and Viswanath in [10], in the case of a Poisson bivector on a manifold, when $A = TM$ and $[\pi, \pi]_\mu = 0$. We proved it independently, and in the general case, in [33]. To prove (iii), use the relations $\{\mu, \mu\} = 0$, and $\{\mu, \gamma_{\mu,\pi}\} = 0$ which follows from (4.2) and the Jacobi identity. Moreover $\{\{\mu, \pi\}, \{\mu, \pi\}\} = \{\mu, [\pi, \pi]_\mu\}$, whence $\frac{1}{2}\{\gamma_{\mu,\pi}, \gamma_{\mu,\pi}\} + \{\mu, \varphi\} = 0$, and

$$-2\{\gamma_{\mu,\pi}, \varphi\} = \{\{\pi, \mu\}, [\pi, \pi]_\mu\} = [\pi, [\pi, \pi]_\mu]_\mu = 0 \ .$$

Thus the four conditions equivalent to (1.2) are satisfied. □

The square of d_π does not vanish in general,

$$(d_\pi)^2 + [\varphi, \ \cdot \ ]_\mu = 0 \ .$$

A necessary and sufficient condition for $((A, A^*), \mu, \gamma_{\mu,\pi})$ to be a Lie bialgebroid is the *generalized Poisson condition*,

$$d_\mu([\pi, \pi]_\mu) = 0 \ , \tag{4.5}$$

which includes, as a special case, the generalized classical Yang-Baxter equation, and which is equivalent to the conditions to be found in [33], page 74, and in Theorem 2.1 in [36].

A sufficient condition is that π satisfy the *Poisson condition*,

$$[\pi, \pi]_\mu = 0 \ , \tag{4.6}$$

which generalizes both the classical Yang-Baxter equation and the definition of Poisson bivectors. This condition is satisfied if and only if the graph of π is a Dirac sub-bundle of the standard Courant algebroid, $A \oplus A^*$, the double of the Lie bialgebroid with trivial cobracket, $((A, A^*), \mu, 0)$. (See [14] for the case where $A = TM$, and [35].) The Lie bialgebroid defined by (A, π), where π satisfies (4.6) is called a *triangular Lie bialgebroid* [36].

By Theorem 3.2, a deriving operator for the Courant bracket of the double of the Lie quasi-bialgebroid $((A, A^*), \mu, \gamma_{\mu,\pi}, -\frac{1}{2}[\pi, \pi]_\mu, 0)$ is

$$d_\mu - \partial_\pi + i_\varphi \ ,$$

where ∂_π is the graded commutator $[i_\pi, d_\mu]$, and $\varphi = -\frac{1}{2}[\pi, \pi]_\mu$. In fact [34] [29], ∂_π generates the bracket $[\ ,\ ]_{\mu,\pi}$ of A^*. If π satisfies the Poisson condition (4.6), then $d_\mu - \partial_\pi$ is a deriving operator.

Dually, $((A^*, A), \gamma_{\mu,\pi}, \mu, 0, \psi)$, with $\psi = -\frac{1}{2}[\pi, \pi]_\mu$, is a quasi-Lie bialgebroid, and $((A^*, A), \gamma_{\mu,\pi}, \mu)$ is a Lie bialgebroid if and only if π satisfies equation (4.5).

4.1.2 Twisting of a proto-bialgebroid

The Lie quasi-bialgebroid (A, A^*) and the dual quasi-Lie bialgebroid (A^*, A) are the result of the *twisting* by the bivector π of the Lie bialgebroid with trivial cobracket, $((A, A^*), \mu, 0)$. The operation of twisting, in this general setting of the theory of Lie algebroids, was defined and studied by Roytenberg in [44]. He showed that one can also twist a proto-bialgebroid, $((A, A^*), \mu, \gamma, \varphi, \psi)$, by a bivector π. The result is a proto-bialgebroid defined by $(\mu'_\pi, \gamma'_\pi, \varphi'_\pi, \psi'_\pi)$, where

$$\mu'_\pi = \mu + \pi^\sharp \psi \ , \tag{4.7}$$

$$\gamma'_\pi = \gamma + \gamma_{\mu,\pi} + (\wedge^2 \pi^\sharp)\psi \ , \tag{4.8}$$

$$\varphi'_\pi = \varphi - d_\gamma \pi - \frac{1}{2}[\pi, \pi]_\mu + (\wedge^3 \pi^\sharp)\psi \ , \tag{4.9}$$

$$\psi'_\pi = \psi \ . \tag{4.10}$$

Here $\pi^\sharp \psi$ is the A-valued 2-form on A such that

$$(\pi^\sharp \psi)(x, y)(\xi) = \psi(x, y, \pi^\sharp \xi) \ ,$$

for all $\xi \in \Gamma(A^*)$, and $(\bigwedge^2 \pi^\sharp)\psi$ is the A^*-valued 2-form on A^* such that,

$$((\wedge^2 \pi^\sharp)\psi)(\xi, \eta)(x) = \psi(\pi^\sharp \xi, \pi^\sharp \eta, x) \ ,$$

for all $x \in \Gamma A$, while $(\bigwedge^3 \pi^\sharp)\psi$ is the section of $\bigwedge^3 A$ such that, for ξ, η and $\zeta \in \Gamma(A^*)$,

$$((\wedge^3\pi^\sharp)\psi)(\xi,\eta,\zeta) = \psi(\pi^\sharp\xi, \pi^\sharp\eta, \pi^\sharp\zeta) \ .$$

A computation shows that the tensors introduced above satisfy the relations

$$\begin{aligned} \pi^\sharp\psi &= \{\pi,\psi\} \ , \\ (\wedge^2\pi^\sharp)\psi &= \frac{1}{2}\{\pi,\{\pi,\psi\}\} \ , \\ (\wedge^3\pi^\sharp)\psi &= \frac{1}{6}\{\pi,\{\pi,\{\pi,\psi\}\}\} \ . \end{aligned}$$

These relations are used to prove that $((A,A^*),\mu'_\pi,\gamma'_\pi,\varphi'_\pi,\psi)$ is a proto-bialgebroid.

This proto-bialgebroid is a Lie quasi-bialgebroid if and only if $\psi = 0$, that is, if the initial object itself was a Lie quasi-bialgebroid.

It is a quasi-Lie bialgebroid if and only if $\varphi'_\pi = 0$, that is,

$$\varphi - d_\gamma\pi - \frac{1}{2}[\pi,\pi]_\mu + (\wedge^3\pi^\sharp)\psi = 0 \ . \tag{4.11}$$

We now list the particular cases of this construction that lead to the various integrability conditions to be found in the literature.

(a) Twist of a Lie bialgebroid: $(\mu,\gamma,0,0) \mapsto (\mu,\gamma+\gamma_{\mu,\pi},-d_\gamma\pi - \frac{1}{2}[\pi,\pi]_\mu,0)$. The result is a Lie quasi-bialgebroid, furthermore it is a Lie bialgebroid if and only if the bivector π satisfies the *Maurer-Cartan equation*,

$$d_\gamma\pi + \frac{1}{2}[\pi,\pi]_\mu = 0 \ . \tag{4.12}$$

This condition is satisfied if and only if the graph of π is a Dirac sub-bundle of the Courant algebroid, $A \oplus A^*$, the double of the Lie bialgebroid $((A,A^*),\mu,\gamma,0,0)$ [35]. A necessary and sufficient condition for $((A,A^*),\mu,\gamma+\gamma_{\mu,\pi})$ to be a Lie bialgebroid is the weaker condition, $d_\mu(d_\gamma\pi + \frac{1}{2}[\pi,\pi]_\mu) = 0$.

If the cobracket γ of (A,A^*) is trivial, to $(\mu,0,0,0)$ there corresponds the quadruple $(\mu,\gamma_{\mu,\pi},-\frac{1}{2}[\pi,\pi]_\mu,0)$: this is the case studied in Section 4.1.1. We know that the result is a Lie quasi-bialgebroid, and it is a Lie bialgebroid if and only if π satisfies the Poisson condition (4.6), and that $((A,A^*),\mu,\gamma_{\mu,\pi})$ is a Lie bialgebroid if and only if the bivector π satisfies the generalized Poisson condition (4.5).

If the bracket μ of (A,A^*) is trivial, to $(0,\gamma,0,0)$ there corresponds the quadruple $(0,\gamma,-d_\gamma\pi,0)$, which gives rise to a Lie bialgebroid if and only if

$$d_\gamma\pi = 0, \tag{4.13}$$

which means that the bivector π on A is closed, when considered as a 2-form on A^*.

(b) Twist of a Lie quasi-bialgebroid: $(\mu,\gamma,\varphi,0) \mapsto (\mu,\gamma+\gamma_{\mu,\pi},\varphi'_\pi,0)$, where $\varphi'_\pi = \varphi - d_\gamma\pi - \frac{1}{2}[\pi,\pi]_\mu$. The result is a Lie quasi-bialgebroid, furthermore it is a Lie bialgebroid if and only if the bivector π and the 3-vector φ satisfy the *quasi-Maurer-Cartan equation*,

$$d_\gamma\pi + \frac{1}{2}[\pi,\pi]_\mu = \varphi \ . \tag{4.14}$$

A necessary and sufficient condition for the pair $((A, A^*), \mu, \gamma + \gamma_{\mu,\pi})$ to be a Lie bialgebroid is the weaker condition, $d_\mu \varphi'_\pi = 0$.

Assume that the cobracket γ of (A, A^*) is trivial. Then, in order for $(\mu, 0, \varphi, 0)$ to define a Lie quasi-bialgebroid, the 3-vector φ must satisfy $\{\mu, \varphi\} = 0$. In this case, condition (4.14) reduces to

$$\frac{1}{2}[\pi, \pi]_\mu = \varphi \ , \tag{4.15}$$

which is a *quasi-Poisson* condition, analogous to (1.1).

(c) Twist of a quasi-Lie bialgebroid: $(\mu, \gamma, 0, \psi) \mapsto (\mu'_\pi, \gamma'_\pi, \varphi'_\pi, \psi)$, where $\mu'_\pi = \mu + \pi^\sharp \psi$, $\gamma'_\pi = \gamma + \gamma_{\mu,\pi} + (\bigwedge^2 \pi^\sharp)\psi$ and $\varphi'_\pi = -d_\gamma \pi - \frac{1}{2}[\pi, \pi]_\mu + (\bigwedge^3 \pi^\sharp)\psi$. The result is a proto-bialgebroid, furthermore it is a quasi-Lie bialgebroid if and only if the bivector π and the 3-form ψ satisfy the *Maurer-Cartan equation with background* ψ or ψ*-Maurer-Cartan equation*,

$$d_\gamma \pi + \frac{1}{2}[\pi, \pi]_\mu = (\wedge^3 \pi^\sharp)\psi \ . \tag{4.16}$$

Assume that the cobracket γ of (A, A^*) is trivial. Then, in order for $(\mu, 0, 0, \psi)$ to define a quasi-Lie bialgebroid, the 3-form ψ must be d_μ-closed. In this case, condition (4.16) reduces to the *Poisson condition with background* ψ or ψ*-Poisson condition*,

$$\frac{1}{2}[\pi, \pi]_\mu = (\wedge^3 \pi^\sharp)\psi \ , \tag{4.17}$$

to be found in [42], [25] and [48].

We shall now consider in greater detail two particular cases of the above construction of a Lie quasi-bialgebroid from a given Lie quasi-bialgebroid equipped with a bivector.

4.1.3 Lie quasi-bialgebras and r-matrices

When the base manifold of a Lie algebroid is a point, it reduces to a Lie algebra, $\mathfrak{g} = (F, \mu)$. An element in $\bigwedge^2 F$ can be viewed as a $\bigwedge^2 F$-valued 0-cochain on $\mathfrak{g}$. The *triangular* r*-matrices* are those elements r in $\bigwedge^2 F$ that satisfy $[r, r]_\mu = 0$. Let us explain why the twisting defined by a bivector generalizes the operation of *twisting* defined on Lie bialgebras, and more generally on Lie quasi-bialgebras, by Drinfeld [15], and further studied in [28] and [8].

In this case, formula (4.1) reduces to

$$[\xi, \eta]_{\mu,r} = -(d_\mu r)(\xi, \eta) \ . \tag{4.18}$$

Here $d_\mu r$ is the Chevalley-Eilenberg coboundary of r, a 1-cochain on $\mathfrak{g}$ with values in $\bigwedge^2 \mathfrak{g}$. This formula is indeed that of the cobracket on F, obtained by twisting a Lie bialgebra with vanishing cobracket by an element $r \in \bigwedge^2 F$ (see [15] [28]). Formulas (4.3) and (4.4) also reduce to the known fomulas.

Then $((F, F^*), \mu, -d_\mu r)$ is a Lie bialgebra if and only if $d_\mu [r, r]_\mu = 0$, *i.e.*, if and only if r satisfies the *generalized classical Yang-Baxter equation*. A sufficient

condition is that r satisfy the *classical Yang-Baxter equation*, $[r,r]_\mu = 0$, in which case r is a triangular r-matrix.

In this purely algebraic case, the Courant bracket of $F \oplus F^*$ is skew-symmetric, and therefore is a true Lie algebra bracket. It satisfies

$$[x,\xi] = -[\xi,x] = -i_\xi d_\gamma x + i_x d_\mu \xi = -ad^{*\gamma}_\xi x + ad^{*\mu}_x \xi \ ,$$

and therefore coincides with the bracket of the *Drinfeld double*.

A deriving operator for the Lie bracket of the Drinfeld double of a Lie proto-bialgebra $((F,F^*),\mu,\gamma,\varphi,\psi)$ is $d_\mu - \partial_\gamma + i_\varphi + e_\psi$, where d_μ (resp., ∂_γ) is the generalization of the Chevalley-Eilenberg cohomology (resp., homology) operator of (F,μ) (resp., (F^*,γ)) to the case where the bracket μ (resp., γ) does not necessarily satisfy the Jacobi identity.

4.1.4 Tangent bundles and Poisson bivectors

When $A = TM$, the tangent bundle of a manifold M, a section π of $\bigwedge^2 A$ is a bivector field on M. Let μ_{Lie} be the function defining the Lie bracket of vector fields, and more generally the Schouten bracket of multivector fields. The associated differential is the de Rham differential of forms, which we denote by d. In this case, we denote the bracket of forms, defined by formula (4.1) above, simply by $[\ ,\ ]_\pi$ and the function $\gamma_{\mu,\pi}$ simply by γ_π. Thus $((TM,T^*M),\mu_{\mathrm{Lie}},\gamma_\pi,\varphi,0)$, with $\varphi = -\frac{1}{2}[\pi,\pi]$, is a Lie quasi-bialgebroid, and if $[\pi,\pi] = 0$, *i.e.*, π is a Poisson bivector, then $((TM,T^*M),\mu_{\mathrm{Lie}},\gamma_\pi)$ is a Lie bialgebroid. The bracket $[\ ,\ ]_\pi$ is then the *Fuchssteiner-Magri-Morosi bracket* [17] [40], its extension to forms of all degrees being the *Koszul bracket* [34].

A deriving operator for the Courant bracket of the double, $TM \oplus T^*M$, of the Lie bialgebroid of a Poisson manifold is $\mathrm{d} - \partial_\pi$, where $\partial_\pi = [i_\pi, \mathrm{d}]$ is the *Poisson homology operator*, defined by Koszul and studied by Huebschmann [20], and often called the Koszul-Brylinski operator. Indeed, it is well known that the operator ∂_π generates the Koszul bracket of forms. This was in fact the original definition given by Koszul in [34]. This deriving operator is of square 0.

We can also consider the dual object. Whenever π is a bivector field on M, $((T^*M,TM),\gamma_\pi,\mu_{\mathrm{Lie}},0,\psi)$, with $\psi = -\frac{1}{2}[\pi,\pi]$, is a quasi-Lie bialgebroid, which, when π is a Poisson bivector, is the Lie bialgebroid dual to (TM,T^*M).

If M is orientable with volume form ν, a deriving operator for the Courant bracket of the double, $T^*M \oplus TM$, is $d_\pi - \partial_\nu$, where $\partial_\nu = - *^{-1} d\, *$ (here, $*$ is the operator on forms defined by ν). In fact, the operator ∂_ν generates the Schouten bracket of multivector fields [34] [31]. To obtain a deriving operator of square 0, we must add to $\mathrm{d} - \partial_\nu$ the derivation e_{X_ν}, where X_ν is the modular vector field of the Poisson manifold (M,π) associated with the volume form ν. In the non-orientable case, one should introduce densities as in [16]. If π is invertible, with inverse Ω, then $\partial = [i_\Omega, d_\pi]$ generates the Schouten bracket [29] and therefore $d_\pi - \partial$ is a deriving operator of square 0 for the Courant bracket of $T^*M \oplus TM$.

4.2 The Courant bracket of Poisson structures with background

4.2.1 The Courant bracket with background

Let (A, μ) be a Lie algebroid and let ψ be a 3-form on A, a section of $\bigwedge^3 A^*$. Then, as we remarked in Section 4.1.2, $((A, A^*), \mu, 0, 0, \psi)$, is a quasi-Lie bialgebroid if and only if the 3-form ψ is d_μ-closed,

$$d_\mu \psi = 0 \ .$$

This is the most general quasi-Lie bialgebroid with trivial cobracket. By definition, the functions μ and ψ satisfy $\{\mu, \mu\} = 0$ and $\{\mu, \psi\} = 0$, so that μ defines a Lie algebroid bracket, but we obtain a Lie bialgebroid if and only if $\psi = 0$.

The bracket of the double $A \oplus A^*$ (in the case of $TM \oplus T^*M$) was introduced by Ševera and Weinstein [48] who called it the modified Courant bracket or the *Courant bracket with background* ψ. This bracket satisfies

$$[x, y] = [x, y]_\mu + i_{x \wedge y} \psi \quad , \quad [x, \xi] = L^\mu_x \xi \quad , \quad [\xi, x] = -i_x d_\mu \xi \quad , \quad [\xi, \eta] = 0 \ ,$$

that is

$$[x + \xi, y + \eta] = [x, y]_\mu + L^\mu_x \eta - i_y d_\mu \xi + i_{x \wedge y} \psi \ .$$

By Theorem 3.2, $d_\mu + e_\psi$ is a deriving operator of the Courant bracket with background ψ.

In the case of a Lie algebra, (F, μ), $[x, \xi] = -[\xi, x] = ad^{*\mu}_x \xi$.

Remark In [8], we considered the case of the most general Lie quasi-bialgebra with trivial cobracket. Similarly, one can consider the Lie quasi-bialgebroids of the form $(\mu, 0, \varphi, 0)$, with $\{\mu, \mu\} = 0$ and $\{\mu, \varphi\} = 0$, and the Courant bracket with background φ, a 3-vector in this case,

$$[x, y] = [x, y]_\mu \quad , \quad [x, \xi] = L^\mu_x \xi \quad , \quad [\xi, x] = -i_x d_\mu \xi \quad , \quad [\xi, \eta] = i_{x \wedge y} \varphi \ ,$$

so that

$$[x + \xi, y + \eta] = [x, y]_\mu + i_{x \wedge y} \varphi + L^\mu_x \eta - i_y d_\mu \xi \ .$$

This case is not dual to the preceding one.

4.2.2 Twisting of the Courant bracket with background

Let $((A, A^*), \mu, 0, 0, \psi)$ be a quasi-Lie bialgebroid with trivial cobracket, where ψ is the background d_μ-closed 3-form. For the corresponding Courant bracket with background, we shall describe the twisting defined as above by a section π of $\bigwedge^2 A$. The twisting of this quasi-Lie bialgebroid, a special case of the that described in Section 4.1.2, yields a proto-bialgebroid whose structural elements depend on μ, ψ and π, and which we shall denote by $(\widetilde{\mu}, \widetilde{\gamma}, \widetilde{\varphi}, \widetilde{\psi})$,

$$\begin{aligned}
\widetilde{\mu} &= \mu + \pi^\sharp \psi \ , \\
\widetilde{\gamma} &= \gamma_{\mu,\pi} + (\wedge^2 \pi^\sharp) \psi \ , \\
\widetilde{\varphi} &= -\frac{1}{2} [\pi, \pi]_\mu + (\wedge^3 \pi^\sharp) \psi \ , \\
\widetilde{\psi} &= \psi \ .
\end{aligned}$$

We have seen in Section 4.1.2(c) that the resulting twisted object is a quasi-Lie bialgebroid if and only if $\widetilde{\varphi} = 0$, *i.e.*, π satisfies the *ψ-Poisson condition* (4.17),

$$\frac{1}{2}[\pi, \pi]_\mu = (\wedge^3 \pi^\sharp)\psi \ .$$

It was shown in [48] that this condition is satisfied if and only if the graph of π is a Dirac sub-bundle in the Courant algebroid with background, $A \oplus A^*$, the double of the quasi-Lie bialgebroid $((A, A^*), \mu, 0, 0, \psi)$. This constitutes a generalization of the property valid in the usual case, reviewed in Section 4.1.1, where $\psi = 0$ and condition (4.17) reduces to the usual Poisson condition.

The associated derivations, on $\Gamma(\bigwedge^\bullet A^*)$ and on $\Gamma(\bigwedge^\bullet A)$, are

$$\begin{array}{rcl} d_{\widetilde{\mu}} & = & d_\mu + i_{\pi^\sharp \psi} \ , \\ d_{\widetilde{\gamma}} & = & [\pi, \ \cdot \ ]_\mu + i_{(\bigwedge^2 \pi^\sharp)\psi} \ . \end{array}$$

Because $\frac{1}{2}\{\widetilde{\mu}, \widetilde{\mu}\} = -\{\widetilde{\gamma}, \widetilde{\psi}\}$, the derivation $d_{\widetilde{\mu}}$ does not have vanishing square in general. On the other hand, whenever π satisfies the ψ-Poisson condition, $d_{\widetilde{\gamma}}$ is a true differential and $\widetilde{\gamma}$ defines a true Lie bracket on $\Gamma(A^*)$, and a true Gerstenhaber bracket on $\Gamma(\bigwedge^\bullet A^*)$, the modified Koszul bracket.

We now consider the Courant bracket of the associated double, the π-twisted Courant bracket with background ψ. The mixed terms are $[x, \xi] = -i_\xi d_{\widetilde{\gamma}} x + L_x^{\widetilde{\mu}} \xi$ and $[\xi, x] = L_\xi^{\widetilde{\gamma}} x - i_x d_{\widetilde{\mu}} \xi$, therefore

$$[x, y] = [x, y]_\mu + (\pi^\sharp \psi)(x, y) \ , \tag{4.19}$$

$$[x, \xi] = -i_\xi [\pi, x]_\mu - (\pi^\sharp \psi)(x, \pi^\sharp \xi) + i_x d_\mu \xi + i_{x \wedge \pi^\sharp \xi} \psi + d_\mu < \xi, x > \ , \tag{4.20}$$

$$[\xi, x] = i_\xi [\pi, x]_\mu + (\pi^\sharp \psi)(x, \pi^\sharp \xi) + [\pi, < \xi, x >]_\mu - i_x d_\mu \xi - i_{x \wedge \pi^\sharp \xi} \psi \ , \tag{4.21}$$

$$[\xi, \eta] = [\xi, \eta]_{\mu, \pi} + i_{\pi^\sharp \xi \wedge \pi^\sharp \eta} \psi \ . \tag{4.22}$$

In particular, for $\psi = 0$, we obtain the Courant bracket of the double of the twist by π of the Lie bialgebroid with trivial cobracket, $((A, A^*), \mu, 0)$, considered in Section 4.1.1. Therefore, whenever π satisfies the generalized Poisson condition (4.5), the above formulas yield the Courant bracket of the double of the Lie bialgebroid $((A, A^*), \mu, \gamma_{\mu, \pi})$. In the purely algebraic case, we recover the Drinfeld double of a coboundary Lie bialgebra, defined by r, an r-matrix solution of the generalized Yang-Baxter equation. Setting $\underline{r}(\xi) = i_\xi r$, and using the relation $i_x d_\mu \xi = ad_x^{*\mu} \xi$, we obtain

$$\begin{array}{l} [x, y] = [x, y]_\mu \ , \\ [x, \xi] = -[\xi, x] = -\underline{r}(ad_x^{*\mu} \xi) + ad_x^\mu (\underline{r} \xi) + ad_x^{*\mu} \xi \ , \\ [\xi, \eta] = ad_{\underline{r} \xi}^{*\mu} \eta - ad_{\underline{r} \eta}^{*\mu} \xi \ . \end{array}$$

To conclude, we prove a property of the Lie bracket defined by $\widetilde{\gamma}$ on $\Gamma(A^*)$.

Proposition 4.1 *If, π satisfies the ψ-Poisson condition, the mapping $\pi^\sharp$ is a morphism of Lie algebroids fom A^* with the Lie bracket* (4.22) *to A with the Lie bracket* $[\ ,\ ]_\mu$.

Proof It is clear that the anchor of A^* is $\rho_A \circ \pi^\sharp$. To prove that $\pi^\sharp$ satisfies

$$\pi^\sharp[\xi, \eta] = [\pi^\sharp\xi, \pi^\sharp\eta]_\mu \ , \tag{4.23}$$

for all ξ and $\eta \in \Gamma(A^*)$, we recall the relation,

$$\pi^\sharp[\xi, \eta]_{\mu,\pi} - [\pi^\sharp\xi, \pi^\sharp\eta]_\mu = \frac{1}{2}[\pi, \pi]_\mu(\xi, \eta) \ , \tag{4.24}$$

proved in [33]. In view of (4.22), where the bracket of A^* is expressed in terms of $[\ ,\ ]_{\mu,\pi}$ and ψ, and of the equality,

$$((\wedge^3\pi^\sharp)\psi)(\xi, \eta) = -\pi^\sharp(i_{\pi^\sharp\xi\wedge\pi^\sharp\eta}\psi) \ ,$$

we see that, when π satisfies the ψ-Poisson condition (4.17), equation (4.23) follows from equation (4.24). $\square$

Conclusion

In the preceding discussion, we have encountered various weakenings and generalizations of the usual notions of Lie bialgebra, Lie algebroid and Poisson structure that have appeared in the literature, starting with Drinfeld's semi-classical limit of quasi-Hopf algebras, and up to the recent developments due in great part to Alan Weinstein, his co-workers and his former students. We hope to have clarified the relationships and properties of these structures.

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